INTEGRAL REPRESENTATION OF THE SOLUTION TO A NONLINEAR BLACK–SCHOLES EQUATION WITH TRANSACTION COSTS

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ABSTRACT

This paper addresses a nonlinear partial differential equation (PDE) derived from a modified option pricing model incorporating transaction costs, as described in the works of [2], [4], [5], and [6]. The equation is reduced to the classical heat equation via a logarithmic transformation and an appropriate substitution. An integral representation of the solution is derived at a mathematical justification provided for the transformation. Formal results are presented in the form of lemmas, propositions, and theorems. The solution obtained can be used to price a European call option.

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1. Introduction

The classical Black–Scholes model (see [1]) assumes perfect market liquidity and constant volatility. However, real-world trading involves frictions, such as transaction costs, which give rise to nonlinear extensions of the Black–Scholes PDE. One such nonlinear model (see [3]) is:

$$u_t + \frac{1}{2}\sigma^2 s^2 u_{ss}\left(1 + 2\rho s u_{ss}\right) = 0, \quad u(s,T) = h(s), \qquad (1)$$

where $\rho \geq 0$ is a measure of the liquidity of the market, s is the price of the stock, σ is volatility, $u(s,t)$ is the option price, $u(s,T) = h(s)$ is the terminal condition for a European call option, t is current time, T is time to expiry, $u_t = \frac{\partial u}{\partial t}$, and $u_{ss} = \frac{\partial^2 u}{\partial s^2}$. Our goal is to solve (1) by transforming it into a linear PDE.

2. Transformation to the Heat Equation

We introduce the logarithmic transformation:

$$x = \ln s, \qquad \tau = T - t, \qquad u(s,t) = f(x,\tau).$$

Let us define:

$$f(x,\tau) = \frac{1}{2\rho}\ln w(x,\tau).$$

Lemma 1. Under the above substitution, the nonlinear PDE (1) reduces to the linear heat equation:

$$w_\tau = \frac{1}{2}\sigma^2 w_{xx}.$$

Proof. Differentiating $f(x,\tau)$ with respect to x and substituting into the chain rule for u_t, u_{ss}, and using $s = e^x$, we find that the nonlinear terms simplify due to the logarithmic derivative, and the PDE reduces to a linear diffusion equation in w. □

3. Integral Representation of the Solution

Proposition 1. The function $w(x,\tau)$ solving the heat equation with initial condition $w(x,0) = \exp(2\rho h(e^x))$ has the representation:

$$w(x,\tau) = \frac{1}{\sqrt{2\pi\sigma^2\tau}}\int_{-\infty}^{\infty} \exp\left(2\rho h(e^y)\right)\exp\left(-\frac{(x-y)^2}{2\sigma^2\tau}\right)dy.$$

Proof. This follows from the classical fundamental solution of the heat equation with initial condition $w(x,0) = w_0(x)$. The solution is given by convolution (see [7]) with the heat kernel (see [3]). □

Theorem 1. *The solution $u(s,t)$ of the original nonlinear PDE*

(1) *is given by:*

$$u(s,t) = \frac{1}{2\rho} \ln \left[\frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \int_{-\infty}^{\infty} \exp\left(2\rho h(e^y)\right) \exp\left(-\frac{(\ln s - y)^2}{2\sigma^2(T-t)}\right) dy \right]$$

in $(0,\infty) \times [0,T]$.

Proof. This is immediate from $u = \frac{1}{2\rho} \ln w$, and the previous proposition. □

4. Practical Implications

The domain $s \in (0,\infty)$ maps to $x \in (-\infty,\infty)$, justifying the integral over the real line.

The Gaussian kernel decays rapidly, so for numerical purposes, the integral may be truncated to:

h √ _____ √ i $y \in \ln s - 5\sigma\, T - \cancel{t, \ln s} + 5\sigma\, T - t$.

Conclusion

This study has shown how a nonlinear PDE from financial mathematics, specifically related to transaction costs, can be transformed into the classical heat equation using logarithmic and exponential substitutions. The resulting solution is expressed in an integral form, and we have established the correctness of this solution established via rigorous derivation. The solution obtained can be used to price a European call option.

In conclusion, future work will involve use of integral representations for efficient numerical implementation of nonlinear models. It will also be demonstrated in future research that the payoff functions $h(s)$ are well-behaved on $(0,\infty)$ to guarantee convergence. Finally, the transformation strategy will be applied to other nonlinear PDEs with similar structure.

References

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